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The metric and gravitational mass of interacting charged particles at rest

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Abstract. An axisymmetric static system of charged point particles is investigated in the framework of general relativity. The particles are located on the axis of symmetry; the Einstein–Maxwell equations are solved in second approximation. Equilibrium is maintained by means of 'struts' connecting neighbouring particles. Assumptions are made which make the solution unique. The mass of the whole system and the gravitational mass of individual particles are defined.

We find that the change in gravitational mass of an individual particle with charge Q due to the presence of the other particles is equal to $-\frac{1}{2}Q\phi$ where ϕ is the electric potential produced by the remaining particles on the location of the particle. This result, which we expect to hold also if the system does not have axial symmetry, should be obtainable from Einstein's linearized theory and the above mentioned assumptions.

1. The exact equations

The line element for any axisymmetric electrostatic field can be written in the form

$$ds^2 = e^{2(\nu-\lambda)}(dr^2+dz^2)+r^2\,e^{-2\lambda}\,d\varphi^2-e^{2\lambda}\,dt^2 \tag{1}$$

and the Einstein–Maxwell equations are equivalent to (Majumdar 1947):

$$\Delta\phi \equiv \phi_{rr}+\phi_{zz}+\frac{1}{r}\phi_r = 2\phi_r\lambda_r+2\phi_z\lambda_z \tag{2a}$$

$$\Delta\lambda \equiv \lambda_{rr}+\lambda_{zz}+\frac{1}{r}\lambda_r = \frac{1}{2}\kappa\,e^{-2\lambda}(\phi_r^2+\phi_z^2) \tag{2b}$$

$$\nu = \int r\{[\tfrac{1}{2}\kappa\,e^{-2\lambda}(\phi_z^2-\phi_r^2)+\lambda_r^2-\lambda_z^2]\,dr+[-\kappa\,e^{-2\lambda}\phi_r\phi_z+2\lambda_r\lambda_z]\,dz\}, \tag{3}$$

where $\kappa = 8\pi$, ϕ is the electrostatic potential and Δ is the Laplace operator. All functions depend on r and z only. Subscripts r and z, with or without comma, denote partial derivatives; eg, $\phi_r = \partial\phi/\partial r$, $\lambda_{2,z} = \partial\lambda_2/\partial z$, etc. Units are chosen so that the speed of light and the gravitational constant are both unity. Except for the appendixes, rational units are used.

The integrand of ν in (3) is an exact differential and ν is therefore independent of the path of integration. We will consider solutions of (1), (2), (3) which correspond to particles on the z axis. It follows then from (3) that $\nu(0, z)$ is constant between any two neighbouring particles. We generally will have

$$\nu(0, z) \neq 0, \tag{4}$$

ie, space on such a part of the z axis is not elementary flat. The corresponding stress singularity is commonly interpreted (Synge 1964, p 313, Robertson and Noonan 1968) as a strut which is required in order to keep the particles at rest. We of course expect that the particles will not stay at rest unless restraining forces are applied. The absolute magnitude of the restraining force between a particle located at $z = z_i$ and the remaining particles of a system is given by

$$\tfrac{1}{4}\,|\nu(0, z_i-\epsilon)-\nu(0, z_i+\epsilon)| \tag{5}$$

where $|\epsilon|$ is an arbitrary small number. It is not always possible to distinguish, in the restraining force on a particle, an electrostatic and a gravitational component. To split the integrand in (3) into two parts, one consisting of all terms containing ϕ, for example, would lead to two integrals whose integrands might not be exact differentials.

2. The approximated equations

Where space–time is almost flat we will have $|\phi|, |\lambda|, |\nu| \ll 1$, and individual terms on the right-hand side of (2) will be much smaller than terms on the left-hand side. To solve (2) by a method of successive approximations, we therefore proceed as Das *et al* (1961) and write

$$\phi = \phi_0 + k\phi_1 + k^2\phi_2 + k^3\phi_3 + \dots \tag{6a}$$

$$\lambda = \lambda_0 + k\lambda_1 + k^2\lambda_2 + k^3\lambda_3 + \dots, \tag{6b}$$

where k is a small and dimensionless, but otherwise arbitrary parameter and numerical subscripts indicate the order of the correction terms. Each choice for k will give a different universe. Substituting (6) into (2) and equating terms with the same power of k gives the following sets of equations.

$$\Delta\phi_a = 2\sum_{b=0}^{a} (\phi_{b,r}\lambda_{a-b,r} + \phi_{b,z}\lambda_{a-b,z}) \tag{7a}$$

$$\Delta\lambda_a = 4\pi \sum_{c=0}^{a} D_c \sum_{b=0}^{a-c} (\phi_{b,r}\phi_{a-c-b,r} + \phi_{b,z}\phi_{a-c-b,z}), \tag{7b}$$

where $a = 1, 2, 3, \dots$ and D_c is the coefficient of k^c in the Taylor series expansion of $e^{-2\lambda}$. Defining $\phi_0 = \lambda_0 = 0$ we need for $a = 1$ and 2 only $D_0 = 1$. The right-hand sides of (7) are known at each stage of the approximation. Writing (7) for $a = 1$ and 2 gives

$$\Delta\phi_1 = 0 \tag{8a}$$

$$\Delta\lambda_1 = 0 \tag{8b}$$

and

$$\Delta\phi_2 = 2(\phi_{1,r}\lambda_{1,r} + \phi_{1,z}\lambda_{1,z}) \tag{9a}$$

$$\Delta\lambda_2 = 4\pi(\phi^2_{1,r} + \phi^2_{1,z}). \tag{9b}$$

Equations (9) can be solved for arbitrary ϕ_1 and λ_1 satisfying (8). We find

$$\phi_2 = \phi_1\lambda_1 + \phi_{2\text{-hom}}, \tag{10a}$$

$$\lambda_2 = 2\pi\phi_1^2 + \lambda_{2\text{-hom}}, \tag{10b}$$

where $\phi_{2\text{-hom}}$ and $\lambda_{2\text{-hom}}$ are solutions of the 'homogeneous' equations associated with (9), and are allowed to have singularities wherever ϕ_1 and λ_1 have singularities. The correctness of (10) has been checked by writing known exact solutions of (1), (2), (3) in the form (6); see appendix 2 for example.

Using (6), we can expand the integrand of v in powers of k, and thus obtain the series

$$v = k^2 v_2 + k^3 v_3 + k^4 v_4 + \ldots . \tag{11}$$

The integrand for each v_a is automatically an exact differential. We find

$$v_2 = \int r\{[4\pi(\phi_{1,z}^2 - \phi_{1,r}^2) + (\lambda_{1,r}^2 - \lambda_{1,z}^2)]\,\mathrm{d}r + [-8\pi\phi_{1,r}\phi_{1,z} + 2\lambda_{1,r}\lambda_{1,z}]\,\mathrm{d}z\}, \tag{12}$$

and

$$v_3 = 2\int r\{[-4\pi\lambda_1(\phi_{1,z}^2 - \phi_{1,r}^2) + 4\pi(\phi_{2,z}\phi_{1,z} - \phi_{2,r}\phi_{1,r}) + \lambda_{1,r}\lambda_{2,r} - \lambda_{1,z}\lambda_{2,z}]\,\mathrm{d}r$$
$$+ [4\pi\lambda_1\phi_{1,r}\phi_{1,z} - 4\pi(\phi_{1,r}\phi_{2,z} + \phi_{2,r}\phi_{1,z}) + \lambda_{1,r}\lambda_{2,z} + \lambda_{2,r}\lambda_{1,z}]\,\mathrm{d}z\}. \tag{13}$$

Thus v_2 and v_3 are known if ϕ_a and λ_a are known for $a = 1$ and 2.

3. A system of *n* point charges

We are interested in systems consisting of point particles located on the z axis and therefore take as solution of (8)

$$\phi_1 = (4\pi)^{-1}\sum_i \frac{Q_i}{\rho_i}, \tag{14a}$$

$$\lambda_1 = -\sum_i \frac{M_i}{\rho_i}, \tag{14b}$$

where

$$\rho_i^2 = r^2 + (z - z_i)^2, \qquad i = 1, 2, \ldots, n. \tag{15}$$

The Q_i, M_i, z_i are constants; n is the number of particles, with the ith particle located at $z = z_i$. The solution (14) has no spherical symmetry even if $n = 1$; but deviations from spherical symmetry far away from the singularity are then small and we are then justified to call kM_1 and kQ_1 the mass and charge respectively (see appendix 2). We want to study systems of particles of the mathematically simplest form and therefore consistently omit dipoles or higher multipoles in our expressions for ϕ and λ. Accordingly we choose for (10)

$$\phi_{2\text{-hom}} = \sum_i \frac{A_i}{\rho_i}, \tag{16a}$$

$$\lambda_{2\text{-hom}} = \sum_i \frac{B_i}{\rho_i}, \tag{16b}$$

where A_i and B_i are constants.

In order to find A_i we only have to assume that the charge kQ_i of each particle is conserved, ie, independent of the presence or absence of other particles. The charge contained in a region of space is a well defined quantity which can be calculated as a surface integral (Synge 1964, p 366), and A_i can thus be found.

To find B_i requires several assumptions, as we are going to show now. At large distances from the system of particles, λ approaches the Newtonian potential. Assuming bounded z_i, we therefore interpret the expression (see appendix 2 for an example)

$$-\lim_{\rho\to\infty} \rho\lambda, \tag{17}$$

where $\rho^2 = r^2 + z^2$ as the mass of the system of particles.

Let us first consider a system consisting of two initially massless (ie, massless when far apart or 'free') particles each with charge kQ; ie,

$$n = 2, \qquad M_1 = M_2 = 0, \qquad kQ_1 = kQ_2 = kQ, \qquad z_1 = -z_2. \tag{18}$$

It is an experimental fact that the energy required in moving these charges slowly (ie, adiabatically) from $z = \pm\infty$ to $z = \pm z_1$, is equal to (we can ignore the difference between the true distance and $|z_1 - z_2|$; see (40)–(44))

$$\frac{k^2 Q^2}{4\pi|z_1 - z_2|}. \tag{19}$$

Assuming that the mass equivalent of this energy appears in the mass of the system as defined by (17), we have, using (10*b*) and (14*a*) that

$$-\lim_{\rho\to\infty} \rho\lambda_2 = -\lim \rho\lambda_{2\text{-hom}} = \frac{Q^2}{4\pi|z_1 - z_2|}. \tag{20}$$

Each singularity will, because of symmetry, contribute the same share to the mass of the system. Equations (16*b*) and (20) lead then uniquely to

$$\lambda_{2\text{-hom}} = -\frac{Q^2}{8\pi|z_1 - z_2|}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right). \tag{21}$$

We now modify (18) by assuming $kQ_2 = dkQ = dkQ_1$. We have, for $d \neq 1$, no reflectional symmetry any more and therefore need an additional assumption in order to find $\lambda_{2\text{-hom}}$. Assuming that a charge dkQ at z_2 produces in a neighbourhood of $(0, z_1)$ a $\lambda_{2\text{-hom}}$ which is d times that which a charge kQ produces, we find

$$\lambda_{2\text{-hom}} = -\frac{dQ^2}{8\pi|z_1 - z_2|}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right). \tag{22}$$

If the number of particles is $n > 2$, we add up $\lambda_{2\text{-hom}}$ for all possible pairs of particles. Thus we get for

$$M_i = 0, kQ_i \text{ at } z_i, \qquad i = 1, \ldots, n, \tag{23}$$

that

$$k^2\lambda_{2\text{-hom}} = -\frac{1}{2}\sum_i \frac{kQ_i}{\rho_i} \sum_{j\neq i} \frac{kQ_j}{4\pi|z_i - z_j|} = -\frac{1}{2}\sum_i \frac{kQ_i}{\rho_i}\bar{\phi}_i, \tag{24}$$

where $\bar{\phi}_i$ is the electric potential produced at $(0, z_i)$ by the $n-1$ charges $kQ_1, \ldots, kQ_{i-1}$, $kQ_{i+1}, \ldots, kQ_n$.

If we divide our system of n particles arbitrarily into two subsystems, and if we define the interaction energy by the distribution

$$(4\pi)^{-1}k^2\Delta\lambda_{2\text{-hom}}, \tag{25}$$

we find as a consequence of the assumption above (22) that each subsystem contains half of the energy of interaction of the two subsystems. This result agrees with the standard assumption on the location of the interaction energy in Newtonian gravitational theory (Synge 1971).

Our basic assumption, that the mass equivalent of the electrostatic energy contributes to the mass of a system would not be needed if we could solve time-dependent problems with point particles. This assumption, which is accepted in classical and quantum electrodynamics has also been made in general relativity (see Florides 1962 for a discussion of the case of a charged sphere). Florides, however, assumes that the mass equivalent is twice the mass equivalent obtained from classical electrostatics; this would give an additional factor 2 on the right-hand sides of (21), (22) and (24). I am, however, unwilling to share this assumption for the reason given by Misner and Putnam (1959). Appendix 1 presents both alternatives for the case of a charged sphere.

4. A system of *n* point charges and one point mass

We now investigate a system consisting of $n+1$ particles, n initially massless charges and one uncharged mass:

$$M_i = 0, kQ_i \text{ at } z_i, i = 1, \ldots, n; kM \text{ at } z_M. \tag{26}$$

Defining

$$\rho_M^2 = r^2 + (z - z_M)^2, \tag{27}$$

and following the procedure outlined below (16), we find after a lengthy calculation as approximation for ϕ:

$$k\phi_1 + k^2\phi_2 = \frac{k}{4\pi}\sum_i \frac{Q_i}{\rho_i} + \frac{k^2 M}{4\pi}\sum_i Q_i \frac{\rho_i - \rho_M - |z_i - z_M|}{\rho_i\rho_M|z_i - z_M|}. \tag{28}$$

It is interesting that this expression at $r = 0$, $z = z_M$ is finite but discontinuous. From (10*b*), (14*a*) and (24) we find as approximation for λ

$$k\lambda_1 + k^2\lambda_2 = -\frac{kM}{\rho_M} + \frac{k^2}{8\pi}\left(\sum_i\sum_j \frac{Q_iQ_j}{\rho_i\rho_j} - \sum_i\sum_{j\neq i} \frac{Q_iQ_j}{\rho_i|z_i - z_j|}\right). \tag{29}$$

Let F denote the third-order approximation for the restraining force between the n charges and the uncharged mass. Equation (5) shows that we can find F by integrating (12) and (13) over a small semicircle of radius ϵ and centre at $(0, z_M)$. Using (5), (28) and (29) and taking the limit as ϵ goes to zero, we find

$$F = \tfrac{1}{4}[k^2 v_2(0, z) + k^3 v_3(0, z)]_{z = z_M - \epsilon}^{z = z_M + \epsilon} = [k^3 M\lambda_{2,z}]_{r=0, z=z_M}. \tag{30}$$

In this case, no terms in v_2 and v_3 containing ϕ contribute to F; we therefore can call F the gravitational force acting on the mass kM.

If the mass is far away from the charges, ie, if

$$\min_i |z_i - z_M| \gg \max_{i,j} |z_i - z_j|, \tag{31}$$

then

$$\lambda_2 \simeq -\frac{1}{8\pi}\sum_i \sum_{j\neq i} \frac{Q_i Q_j}{\rho_i |z_i - z_j|}. \tag{32}$$

Equations (32) and (30) yield an expected result, namely that the constraining force F between a distant mass kM and a system of initially massless charges corresponds to—ie, is the negative of ('action equal and opposite reaction')—the force between a mass kM and a mass which is the mass equivalent of the electrostatic energy of the charges. This will also be true if the charges and the distant mass are not located on the z axis.

We get an unexpected result if we locate the mass kM close—but not too close—to one of the charges, say kQ_1. If $|z_1 - z_M|$ satisfies the inequality

$$\min_i |z_1 - z_i| \gg |z_1 - z_M| \gg \left| Q_1 \left(\sum_{j\neq 1} \frac{Q_j}{|z_1 - z_j|} \right)^{-1} \right|, \tag{33}$$

then we find

$$k^2\lambda_2 \simeq \frac{1}{8\pi}\frac{kQ_1}{\rho_1}\sum_{j\neq 1}\frac{kQ_j}{|z_1 - z_j|} = \frac{1}{2}\frac{kQ_1}{\rho_1}\bar{\phi}_1, \tag{34}$$

where $\bar{\phi}_1$ is the electric potential produced at $r = 0$, $z = z_1$ by the $n-1$ charges kQ_2, $\ldots, kQ_n$. This means that the dominant term in the constraining force F corresponds to the force between a mass kM and a mass $-\frac{1}{2}kQ_1\bar{\phi}_1$ located at z_M and z_1 respectively. This result will also hold if the charges and the mass are not located on the z axis; we only have to replace in (33) the expressions $|z_1 - z_M|$, $|z_1 - z_i|$ by the distances between kQ_1 and kM, kQ_1 and kQ_i respectively. It is therefore reasonable to call

$$-\tfrac{1}{2}kQ_1\bar{\phi}_1 \tag{35}$$

the gravitational mass of the charge kQ_1. If, as in the case of electrons and protons, the charge is initially not massless but has a gravitational mass kM_1 such that

$$|kM_1| \ll |kQ_1|, \tag{36}$$

then the gravitational mass will be

$$kM_1 - \tfrac{1}{2}kQ_1\bar{\phi}_1. \tag{37}$$

As an example let us consider an electron, $kM_1 = 2{\cdot}2\times10^{-66}$ s, $|kQ_1| = 4{\cdot}6\times10^{-45}$ s, in the electric potential of the earth. We approximate the electric charge of the earth by a surface charge of $+3\times10^{24}\,e$ (Smith 1963) on a sphere of radius $r = 2\times10^{-2}$ s $=$ mean radius of earth, and of $-3\times10^{24}\,e$ on a sphere of radius $R = 2\times10^{-2}+4\times10^{-6}$ s. The potential close to the surface of the earth is then given by

$$\bar{\phi}_1 = -(4\pi)^{-1}(3\times10^{24})(4{\cdot}6\times10^{-45})\left(\frac{1}{r}-\frac{1}{R}\right) \simeq -10^{-23}. \tag{38}$$

The change in gravitational mass of an electron close to the surface of the earth is therefore

$$-\tfrac{1}{2}kQ_1\bar{\phi}_1 = -\tfrac{1}{2}(-4{\cdot}6\times10^{-45})(-10^{-23}) = -2{\cdot}3\times10^{-68}\text{ s}, \tag{39}$$

which is about 1% of the electron's rest mass.

5. Discussion

The main result of this work is contained in § 4. The truncated series of (6) defined by (28) and (29) are good approximations only where $|\phi_1|$, $|\lambda_1| \ll 1$, which is not the case close to the singularities representing particles. The calculation of F in (30) is not affected by this. To let the radius ϵ of the semicircle go to zero allows us to put F in a simple form; a finite ϵ, corresponding to the radius of the particle gives precisely the same value for F.

The singularities, however, give rise to difficulties in the calculation of distances. The distance $s_{i,i+1}$ between the ith and the $i+1$st particle is

$$s_{i,i+1} = \left| \int_{z=z_i}^{z_{i+1}} e^{\nu-\lambda}\, \mathrm{d}z \right|. \tag{40}$$

This integral might not exist. We therefore replace the singularities by particles of finite size. We simply have to continue ϕ, λ, ν in such a way across a chosen boundary B of the particle that ϕ, λ, ν and their first partial derivatives are continuous across B and stay small inside B (we would have to be more selective in our continuation of ϕ, λ, ν if we should desire positive mass density everywhere inside B). Let us take for B of the ith particle the 'sphere'

$$\rho_i^2 \equiv r^2 + (z-z_i)^2 = a_i^2 \tag{41}$$

where a_i is a constant such that

$$|\lambda_1|, |\phi_1| \ll 1 \qquad \text{for } \rho_i = a_i. \tag{42a}$$

We assume, furthermore, that a_i satisfies

$$a_i \ll \min(|z_j - z_k|, |z_j - z_M|). \tag{42b}$$

If we now replace $\lambda_1(r, z)$ (similarly for λ_2, ϕ_1, etc) inside B by, say,

$$\lambda_1(r, z) \cos^4 \left(\frac{\pi}{a_i^2} [a_i^2 - r^2 - (z-z_i)^2] \right) \tag{43}$$

we have λ_1 smooth at B, bounded inside B and $\lambda_1(0, 0) \to 0$. In this way we can achieve $|\lambda_k|, |\nu_k| \ll 1$ everywhere and find then from (40)

$$s_{i,i+1} \simeq |z_{i+1} - z_i|. \tag{44}$$

Treating particles as singularities might also be the reason why attempts to find expressions for ϕ_3 and λ_3 similar to those for ϕ_2 and λ_2 (10) failed.

We used a particle of mass kM (26), ν_3 and F (30) to define the gravitational mass of charged particles. We could have used instead, as indicated in appendix 1,

$$g_{44} = -\exp 2(k\lambda_1 + k^2\lambda_2)$$

and the equations of geodesics (see Rosen 1949). Using the acceleration thus obtained as a measure of gravitational mass would give the same result (35).

Einstein's linearized theory, applied to the system (23), would give an approximation for the metric from the energy–momentum tensor $T_{\mu\nu}$ of classical electrostatics. This $T_{\mu\nu}$ would be quadratic in the charges (ie, electric field strength) and so would the resulting metric tensor (see, for example, (A.6) and (A.10)). We find, therefore, that for the system

(23), $-2k^2\lambda_2$ of (29) equals $1+g_{44}$ of the linearized theory and our result (35) could therefore be obtained from this theory.

In physical reality we rarely find charges kept at rest by struts. Charges usually arrange themselves in such a way that $v(0, z) = 0$. For example, $v_2(0, z) = 0$ everywhere for $+Q$ at $z_1 = 0$, $-4Q$ at $z_2 = -z_3 \equiv z$; a charge in the centre of a symmetric (invariant under $z \to -z$) charge distribution needs no support. A small electric field, resulting in a small v, does, however, not imply a small electric potential ϕ. Variations of ϕ between different galaxies might produce observable changes in the gravitational mass of electrons.

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Appendix 1

The suffixes u, v take the values 1, 2, 3, 4 and α takes the values 1, 2, 3. The gravitational field of a charged sphere of radius a and charge e is, for $r \geqslant a$, given by

$$ds^2 = \left(1-\frac{2\mu}{r}+\frac{e^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2+\sin^2\theta\, d\varphi^2) - \left(1-\frac{2\mu}{r}+\frac{e^2}{r^2}\right) dt^2. \quad (A.1)$$

Florides, in order to arrive at a physical interpretation of the constant μ, makes the following assumptions:

(i) Let m be the 'bare mass' of the uncharged sphere.

(ii) The density of energy for $r \geqslant a$ is given by $\mathcal{T}_4^4$ of Møller's energy–momentum pseudotensor $\mathcal{T}_v^u$.

(iii) The Newtonian potential calculated for this energy density and m accelerates slow test particles in accordance with Newtonian mechanics.

Comparing the acceleration thus obtained with the acceleration obtained from the equations of geodesics for (A.1), Florides obtains

$$\mu = m+\frac{e^2}{a}. \quad (A.2)$$

A different result is obtained by the following procedure. We use Einstein's linearized theory to calculate g_{44} and compare this g_{44} with the exact g_{44} of (A.1). Equating coefficients of the same power of r then will lead to our interpretation of μ. We do not have to equate accelerations or to calculate other components of the metric tensor. Acceleration of slow test particles is governed by g_{44} alone; furthermore, g_{44} behaves like an invariant under transformations of the space-like variables.

According to Einstein's linearized theory we have, assuming spherical symmetry,

$$\Delta g_{44}(r) = r^{-2}(r^2 g_{44,r})_{,r} = -16\pi(T_{44}-\eta_{44}T_u^u) = 8\pi(T_4^4 - T_\alpha^\alpha). \quad (A.3)$$

We raise and lower suffixes with

$$\eta_{uv} = \eta^{uv} \equiv \text{diag}(1, 1, 1, -1), \quad (A.4)$$

and the energy–momentum tensor T^{uv} (of electric field + particle), when expressed in Cartesian coordinates x, y, z, t, has to satisfy

$$T^{uv}{}_{,v} = 0 \qquad \text{everywhere.} \tag{A.5}$$

For $r \geqslant a$ we have from classical electrostatics

$$T^4_4 = -T^\alpha_\alpha = -\frac{e^2}{8\pi r^4}. \tag{A.6}$$

Thus for $r \geqslant a$,

$$\Delta g_{44} = -\frac{2e^2}{r^4} \Rightarrow g_{44} = -1 - \frac{e^2}{r^2} + \frac{\text{constant}}{r}. \tag{A.7}$$

At this stage, ie, for T^u_v of (A.6) only, we have $g_{44,r} = 0$ for $r = a$ and find therefore at this stage constant $= 2e^2/a$.

We now need some information about T_{uv} for $r \leqslant a$. In agreement with Florides' first assumption, we define the mass of the uncharged sphere by

$$4\pi \int_0^a T_{44} r^2 \, \mathrm{d}r \equiv m. \tag{A.8}$$

To get the corresponding integral for T^α_α, we follow Heitler (1954, p 420) and assume that for the electron, including his field

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T_{4u} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

is a four-vector (the energy–momentum vector of the electron). This implies, as Heitler shows (in his proof replace $\langle T_{uv} \rangle$ by T_{uv})

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T_{xx} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = 0 \Rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (T_{xx} + T_{yy} + T_{zz}) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T^\alpha_\alpha \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = 0 \Rightarrow 4\pi \int_0^\infty T^\alpha_\alpha r^2 \, \mathrm{d}r = 0$$

$$\Rightarrow 4\pi \int_0^a T^\alpha_\alpha r^2 \, \mathrm{d}r = -4\pi \int_a^\infty T^\alpha_\alpha r^2 \, \mathrm{d}r = -\frac{e^2}{2a}. \tag{A.9}$$

This integral represents non-Maxwellian stresses (Poincaré binding force) which are essential for the stability of the electron. Using (A.3), (A.7), (A.8), (A.9) we find

$$g_{44} = -1 + \frac{2m}{r} + \frac{e^2}{ar} - \frac{e^2}{r^2}. \tag{A.10}$$

Comparing this with g_{44} of (A.1) gives

$$\mu = m + \frac{e^2}{2a}. \tag{A.11}$$

We would have obtained Florides' result (A.2) if we would have ignored the contribution of (A.9). But to ignore this term would mean that we work with a T_{uv} which does not satisfy

$$T^{uv} n_v = \text{continuous} \tag{A.12}$$

across any two-space (in our case $r = a$) with normal vector n_ν. Our result (A.11) is consistent with the classical theory of the electron (see Rohrlich 1965, p 125, who also discusses the structure of the electron).

Let us now consider two charges connected by a strut as a problem in classical electrostatics. Because of the previous discussion, we only have to consider cross-terms in $T_{\mu\nu}$ (ie, terms containing the electric field of both particles).

To show that the integral over the Maxwellian stresses is cancelled by the stresses in the strut is then easy (Whittaker 1935).

Appendix 2

The coordinate transformation which brings the Reissner Nordstrøm metric (A.1) into the form (1) is given by (in this appendix we use R, φ, z, t to denote the cylinder coordinates of (1))

$$\left.\begin{aligned} r &= \tfrac{1}{2}(R_1+R_2+2\mu) \\ \theta &= \cos^{-1}\frac{R_2-R_1}{2(\mu^2-e^2)^{1/2}} \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} z &= (r-\mu)\cos\theta \\ R &= (r^2-2\mu r+e^2)^{1/2}\sin\theta \end{aligned}\right. \tag{A.13}$$

$$\varphi = \varphi, \qquad t = t,$$

where

$$\begin{aligned} R_1^2 &= [z-(\mu^2-e^2)^{1/2}]^2+R^2 \\ R_2^2 &= [z+(\mu^2-e^2)^{1/2}]^2+R^2. \end{aligned} \tag{A.14}$$

We find

$$\begin{aligned} \phi &= -\frac{e}{r} \\ e^{2\lambda} &= 1-\frac{2\mu}{r}+\frac{e^2}{r^2} \\ e^{2\lambda-2\nu} &= \left(1-\frac{2\mu}{r}+\frac{e^2}{r^2}\right)\cos^2\theta+\left(1-\frac{\mu}{r}\right)^2\sin^2\theta. \end{aligned} \tag{A.15}$$

We see that, in accordance with (17),

$$-\lim_{R\to\infty} R\lambda = -\lim_{z\to\infty} z\lambda = \mu. \tag{A.16}$$

We now expand ϕ and λ in a series according to (6). With

$$\rho^2 \equiv z^2+R^2; \qquad r = \rho+\mu+\text{terms } O(\mu^2-e^2) \tag{A.17}$$

we find

$$\phi = \frac{-e}{\rho}\left(1-\frac{\mu}{\rho}\right)+\text{terms } O(e^{3-p}\mu^p); \qquad p = 0, 1, 2, 3 \tag{A.18a}$$

and

$$\lambda = -\frac{\mu}{\rho}+\frac{e^2}{2\rho^2}+\text{terms } O(e^{3-p}\mu^p). \tag{A.18b}$$

Thus

$$k\phi_1 = -\frac{e}{\rho}, \qquad k^2\phi_2 = \frac{e\mu}{\rho^2}, \tag{A.19a}$$

$$k\lambda_1 = -\frac{\mu}{\rho}, \qquad k^2\lambda_2 = \frac{e^2}{2\rho^2}. \tag{A.19b}$$

Equation (10) (with $\phi_{2-\mathrm{hom}} = \lambda_{2-\mathrm{hom}} = 0$) is, as expected, satisfied. We furthermore conclude that ϕ_1 and λ_1 of (14) for $n = 1$ are not inconsistent with spherical symmetry; ie, properly chosen higher-order terms can produce a metric with spherical symmetry.

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